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HOMOGENEOUS FUNCTIONALS AND STRUCTURAL OPTIMIZATION PROBLEMS

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Abstract—Extremal problems with homogeneous functionals with one or several constraints are investigated. On the basis of general properties of the optimal and quasioptimal solutions of these problems, optimization problems of elastic structures are considered and discussed.

INTRODUCTION

The paper deals with extremal problems with homogeneous functionals and their applications to structural optimization problems with one or several constraints. From the point of view of applications, the optimization problems of elastic structures with multiple restrictions on mechanical behaviour (multi-purpose problems) are of particular interest. Necessary optimality conditions derived in these problems lead to complicated nonlinear boundary value problems with unknown coefficients to be determined from isoperimetric conditions. The presence of multiple parameters makes numerical solutions nonuniversal and hence nonefficient.

Problems of this type were first considered by Prager and Shield[1] and developed in the series of works[2–8]. Problems with the continuum set of constraints (minimax problems) were investigated in[9–11].

In the present paper, by using the homogeneity of a number of functionals having mechanical meaning, it is shown that the solutions of dual structural optimization problems differ from each other by only a multiplier. For extremal problems with multiple constraints the method of the so-called quasioptimal solutions is proposed. This method enables one, on the basis of the solutions of optimal problem only with one constraint, to construct the admissible solutions and to obtain two-sided estimates of the minimum value of the minimized functional. In the examined example of a solid elastic beam with three different constraints on structural behaviour it is shown that the quasioptimal solutions differ from the optimal by the value of minimized functional less than 2.1% and that this estimate is independent of the problem parameters. In this paper it is also shown that the method of the quasioptimal solutions may be applied to multi-purpose structural optimization problems of various mechanical systems such as trusses, beams, arches and plates with various static, dynamic and buckling constraints.

1. PROBLEMS WITH A SINGLE CONSTRAINT

Let us consider the extremal problem with homogeneous functionals. Let E be a linear space and the set K – a cone in it. This means that if an element $x \in K$, then $\lambda x \in K$, $\lambda = \text{const} > 0$.

Let φ and g be homogeneous functionals defined on E with homogeneity degrees α and β respectively, i.e. for $\lambda > 0$

$$\varphi(\lambda x) = \lambda^{\alpha}\varphi(x), \qquad g(\lambda x) = \lambda^{\beta} g(x).$$

Suppose that these functionals are positive when $x \in K$ and consider the extremal problem

$$\begin{gathered} \min \varphi(x) \\ g(x) \geqslant g_0 > 0 \\ x \in K. \end{gathered} \tag{1.1}$$

It is evident that the stated problem is meaningful if α and β are numbers of the same sign. Otherwise the solution does not exist. For convenience we assume α and β to be positive numbers.

It may easily be shown that if x_0 is the solution of problem (1.1) then $g(x_0) = g_0$. This means that the minimum is attained at the limit of the restriction. In fact, if $g(x_0) > g_0$ then we choose a multiplier $\lambda = [g_0/g(x_0)]^{1/\beta} < 1$. The element λx_0 is admissible for the problem under consideration because $g(\lambda x_0) = \lambda^\beta g(x_0) = g_0$ but owing to $\lambda < 1$, $\alpha > 0$ $\varphi(\lambda x_0) < \varphi(x_0)$ in contradiction with the optimality of x_0.

Now consider the other extremal problem with the same functionals

$$\begin{gathered} \max g(y) \\ \varphi(y) \leqslant \varphi_0 > 0 \\ y \in K. \end{gathered} \tag{1.2}$$

Extremum of this problem is also attained at the limit $\varphi(y_0) = \varphi_0$. We shall prove the validity of the following assertion:

(1) If x_0 is the solution of (1.1) then the element

$$y_0 = \gamma x_0, \qquad \gamma = [\varphi_0/\varphi(x_0)]^{1/\alpha},$$

is a solution of (1.2).

In order to prove this fact consider any element $y \in K$ so that $\varphi(y) \leqslant \varphi_0$. We must show that $g(y) \leqslant g(y_0)$. Choosing the multiplier $\chi = [g_0/g(y)]^{1/\beta}$ we can make the element χy admissible for the problem (1.1). In fact, $\chi y \in K$ and $g(\chi y) = g_0$.

Because of the optimality of x_0 we have $\varphi(\chi y) \geqslant \varphi(x_0)$ or $[g_0/g(y)]^{\alpha/\beta}\varphi(y) \geqslant \varphi(x_0)$. Using the conditions $g(x_0) = g_0$ and $\varphi(y) \leqslant \varphi_0$ from the last inequality we get

$$g(y) \leqslant g(x_0)[\varphi(y)/\varphi(x_0)]^{\beta/\alpha} \leqslant g(x_0)[\varphi_0/\varphi(x_0)]^{\beta/\alpha} = g[(\varphi_0/\varphi(x_0))^{1/\alpha}x_0] = g(\gamma x_0) = g(y_0). \tag{1.3}$$

So, assertion (1) is proved. The converse is also true.

(2) If y_1 is the solution of the problem (1.2) then the element

$$x_1 = \mu y_1, \qquad \mu = [g_0/g(y_1)]^{1/\beta}$$

is a solution of (1.1).

The validity of this assertion is proved similarly to the above. Now we show that $\gamma\mu = 1$. Using equalities $g_0 = g(x_0)$ and $\varphi_0 = \varphi(y_1)$, we have

$$\gamma\mu = \left[\frac{\varphi_0}{\varphi(x_0)}\right]^{1/\alpha}\left[\frac{g_0}{g(y_1)}\right]^{1/\beta} = \left[\frac{\varphi(y_1)}{\varphi(x_0)}\right]^{1/\alpha}\left[\frac{g(x_0)}{g(y_1)}\right]^{1/\beta} = \mu^{-1}[\varphi(x_1)/\varphi(x_0)]^{1/\alpha}\mu[g(x_0)/g(x_1)]^{1/\beta} = 1. \tag{1.4}$$

Here $\varphi(x_0) = \varphi(x_1)$ and $g(x_0) = g(x_1)$ because of optimality of x_0 and x_1.

Problems (1.1) and (1.2) will be called dual problems.

(3) If the solution of one of the problems (1.1)–(1.2) exists and is unique, then the solution of the dual problem also exists and is unique, these solutions being connected by the expression $y_0 = \gamma x_0$, $\gamma = [\varphi_0/\varphi(x_0)]^{1/\alpha} = [g_0/g(y_0)]^{-1/\beta}$.

The proof of this assertion is based on the assertions (1) and (2) and the expression (1.4), with the sign of the strict inequality appearing in the chain of the inequalities (1.3) due to the assumption of the solution uniqueness.

Remark 1. Because of the homogeneity of functionals considered the solutions of the problems (1.1), (1.2) x_0 and y_0 can be expressed in the form

$$x_0 = g_0^{1/\beta}x_*, \qquad y_0 = \varphi_0^{1/\alpha}y_* \tag{1.5}$$

where x_*, y_* are the solutions of respective problems when $g_0 = 1$, $\varphi_0 = 1$.

Remark 2. If functionals φ and g have homogeneity degrees of different signs then the problems

$$\begin{array}{cc} \min \varphi(x) & \min g(x) \\ g(x) \leq g_0 < 0 & \varphi(x) \leq \varphi_0 > 0 \\ x \in K & x \in K \end{array}$$

are meaningful and dual. It can be seen with the use of substitution $g_1(x) = 1/g(x)$ then for functionals φ and g with the same signs of homogeneity degrees we obtain the above problems (1.1), (1.2).

2. EXAMPLE 1. OPTIMAL CIRCULAR PLATE WITH CONSTRAINT ON FREQUENCY OF VIBRATION

Free transverse rotationally symmetric vibrations of a circular plate are governed by the equation written in non-dimensional form[12, 7]

$$L_h w = \omega^2 h(r) w(r) r, \qquad r \in (0, 1) \tag{2.1}$$

where

$$L_h = \frac{d}{dr}\left\{ r\left[h^3(r)\left(\frac{d}{dr}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\right) + \frac{d}{dr}(h^3(r))\left(\frac{d^2}{dr^2} + \frac{\nu}{r}\frac{d}{dr}\right)\right]\right\} \tag{2.2}$$

$$\omega^2 = 12(1-\nu^2)\Omega^2 R^2 \rho / E, \qquad h(r) = H(r)/R, \qquad w(r) = W(r)/R. \tag{2.3}$$

Here E, ρ, Ω, R are dimensional quantities: Young's modulus, mass density, frequency of vibrations and radius of the plate, respectively, ω-is non-dimensional frequency, ν is Poisson's ratio, functions $H(r)$, $W(r)$ denote plate thickness and deflection, $h(r)$, $w(r)$ are corresponding non-dimensional quantities. Boundary conditions for simply supported plate are described by the relations

$$[h^3(r)[w'' + \nu w'/r)]_{r=1} = 0, \quad w(1) = 0 \quad [h^3(r)(w'' + w'/r)' + (h^3(r))'(w'' + \nu w'/r)]_{r=0} = 0, \quad w'(0) = 0. \tag{2.4}$$

Next, the non-dimensional volume of the plate v is given by the functional

$$v = \int_0^1 rh(r)\,dr. \tag{2.5}$$

Since the operators L_h and hr are positive and self-adjoint for the first natural frequency the Rayleigh principle can be applied[12, 15]

$$\omega^2(h) = \min_u (L_h u, u)/(hru, u). \tag{2.6}$$

In this expression an admissible function u must satisfy kinematical boundary conditions $u(1) = u'(0) = 0$ and be differenciable. The scalar product $(L_h u, u)$ is given by the integral

$$(L_h u, u) = \int_0^1 h^3(r)[u''^2 + 2\nu u''u'/r + (u'/r)^2]\,r\,dr. \tag{2.7}$$

From the relations (2.5)–(2.7) it is seen that $\omega^2(h)$, $v(h)$ are homogeneous functionals of h with the degrees of homogeneity $\beta = 2$, $\alpha = 1$, respectively.

Consider the optimization problem

$$\begin{gathered} \min v(h) \\ \omega^2(h) \geq \omega_0{}^2 \\ h(r) > 0, \qquad r \in (0, 1). \end{gathered} \tag{2.8}$$

For this problem only a weak minimum is found because, as it can be shown by means of optimal control methods, the strong minimum does not exist.

According to results obtained in Section 1 the next problem is dual to (2.8)

$$\begin{gathered} \max \omega^2(h) \\ v(h) \leq v_0 \\ h(r) > 0, \qquad r \in (0,1). \end{gathered} \tag{2.9}$$

The problem (2.9) was considered and solved numerically in[12] while (2.8) with the supplementary constraint on stiffness under action of static load was treated in[7].

If the solution of (2.8) is represented in the form of (1.5)

$$h_1^0(r) = \omega_0 h^*(r),$$

provided that h^* is the solution of (2.8) with $\omega_0 = 1$, then the solution of the dual problem (2.9) takes the form

$$h_2^0(r) = \frac{v_0}{v(h^*)} h^*(r).$$

Some opinions about equivalence of dual structural optimization problems were expressed earlier by many authors but the strict proof of this fact appears to be new.

It should be noted that many functionals having mechanical meaning such as vibration frequencies, buckling load, static compliance, values of stress or deflection at given points are homogeneous in the structural variable (structural thickness, cross-sectional area, etc.). It is valid for trusses, beams, arches and plates governed by linear differential equations with homogeneous boundary conditions.

3. PROBLEMS WITH MULTIPLE CONSTRAINTS

Let us consider the extremal problem with multiple constraints

$$\begin{gathered} \min \varphi(x) \\ g_i(x) \geq g_i^0 > 0, \qquad i = 1, 2, \dots n. \\ x \in K \end{gathered} \tag{3.1}$$

As before, functionals φ and g_i, $i = 1, 2, \dots n$ are assumed to be positive in the cone K and homogeneous in x with positive degrees of homogeneity α and β_i, $i = 1, 2, \dots n$, respectively.

Now consider the problem (3.1) but only with the single ith constraint. Let the solution of this problem be x_i^0 and the solution of (3.1) with all the constraints x^0. If we calculate the values $g_j(x_i^0)$, $j = 1, 2, \dots n$, then possibly some of the restrictions (3.1) will be not satisfied. Choose now a multiplier γ_i to make the element $\gamma_i x_i^0$ admissible, i.e. satisfying all the constraints. It is easy to see that γ_i may be taken in the form

$$\gamma_i = \max_{j=1,2,\dots n} [g_j^0 / g_j(x_i^0)]^{1/\beta_j}. \tag{3.2}$$

Indeed,

$$g_j(\gamma_i x_i^0) = \gamma_i^{\beta_j} g_j(x_i^0) \geq [(g_j^0/g_j(x_i^0))^{1/\beta_j}]^{\beta_j} g_j(x_i^0) = g_j^0.$$

Because the element $\gamma_i x_i^0$ is admissible but not optimal we have $\varphi(x^0) \leq \varphi(\gamma_i x_i^0)$. On the other hand, $\varphi(x_i^0) \leq \varphi(x^0)$ because the element x_i^0 being the solution of the problem with a single constraint cannot attach to minimized functional value greater than x^0 being the solution of (3.1) with the complete set of constraints. Combining these two inequalities we get

$$\varphi(x_i^0) \leq \varphi(x^0) \leq \varphi(\gamma_i x_i^0).$$

Since i is an arbitrary integer number the following relation is valid

$$\max_{i=1,2,\dots n} \varphi(x_i^0) \leq \varphi(x^0) \leq \min_{i=1,2,\dots n} \varphi(\gamma_i x_i^0). \tag{3.3}$$

The admissible solution $x_q = \gamma_p x_p^0$ which minimizes the right hand relation, we define as quasioptimal solution. According to this definition

$$\varphi(x_q) = \varphi(\gamma_p x_p^0) = \min_{i=1,2,\dots n} \varphi(\gamma_i x_i^0). \tag{3.4}$$

From the last two relationships we obtain

$$1 \leq \varphi(x_q)/\varphi(x^0) \leq \min_{i=1,2,\dots n} \varphi(\gamma_i x_i^0)/\max_{i=1,2,\dots n} \varphi(x_i^0). \tag{3.5}$$

Thus, if the solutions of the extremal problems with a single constraint x_i^0 $i = 1, 2, \dots n$ are known one may calculate the constants γ_i, $i = 1, 2, \dots n$ according to (3.2), construct quasioptimal solution (3.4) and estimate its closeness to the optimal solution by the value of optimized functional.

It is interesting to study the dependence of the upper bound estimate (3.5) on the problem parameters g_i^0, $i = 1, 2, \dots n$. For this reason we express optimal solutions x_i^0 in the form (1.5)

$$x_i^0 = g_i^{0\,1/\beta_i} x_i^*, \qquad i = 1, 2, \dots n. \tag{3.6}$$

Remember that x_i^* is the solution of problem (3.1) with the single ith constraint at $g_i^0 = 1$. Using homogeneity of functionals and the expressions (3.6) for constants γ_i we get

$$\gamma_i = \frac{\max_{j=1,2,\dots n} [g_j^{0\,1/\beta_j} g_j^{-1/\beta_j}(x_i^*)]}{g_i^{0\,1/\beta_i}} = \frac{g_{k_i}^{0\,1/\beta_{k_i}} g_{k_i}^{-1/\beta_{k_i}}(x_i^*)}{g_i^{0\,1/\beta_i}}. \tag{3.7}$$

Here k_i denotes the value of the index j at which maximum of the expression in square brackets is attained.

Now transform (3.5) with the use of (3.6), (3.7)

$$1 \leq \frac{\varphi(x_q)}{\varphi(x^0)} \leq \frac{\min_{i=1,2\dots n} \varphi(\gamma_i x_i^0)}{\max_{i=1,2\dots n} \varphi(x_i^0)} = \min_{\substack{i=1,2\dots n\\ j=1,2\dots n}} \frac{\varphi(\gamma_i x_i^0)}{\varphi(x_j^0)}$$
$$= \min_{\substack{i=1,2,\dots n\\ j=1,2,\dots n}} \left[\frac{g_{k_i}^{0\,\alpha/\beta_{k_i}} g_{k_i}^{-\alpha/\beta_{k_i}}(x_i^*)\varphi(x_i^*)}{g_j^{0\,\alpha/\beta_j}\varphi(x_j^*)}\right] \leq \min_{i=1,2,\dots n} \frac{g_{k_i}^{-\alpha/\beta_{k_i}}(x_i^*)\varphi(x_i^*)}{\varphi(x_{k_i}^*)} \tag{3.8}$$

The last inequality holds because the minimum of the set of terms increases when the overall number of these terms is reduced; in the square brackets only the terms with the indeces $j = k_i$ were retained.

The obtained estimate (3.8) does not contain the parameters g_i^0, but depends upon them as the vector $\mathbf{k} = (k_1, k_2, \dots k_n)$ is defined by the relations between these parameters, see (3.7). To recognize what of the vectors $\mathbf{k} = (k_1, k_2, \dots k_n)$ can be realized it is required to investigate the system of linear inequalities of the following type

$$(g_{k_i}^0/g_{k_i}(x_i^*))^{1/\beta_{k_i}} \geq (g_j^0/g_j(x_i^*))^{1/\beta_j}, \quad \begin{matrix} j = 1, 2, \dots n, j \neq k_i \\ i = 1, 2, \dots n \end{matrix} \quad g_p^{0\,1/\beta_p} > 0, \qquad p = 1, 2, \dots n. \tag{3.9}$$

In the n-dimensional space of variables $g_i^{0\,1/\beta_i}$, $i = 1, 2, \dots n$ the relations (3.9) define the region $G(\mathbf{k})$. If this region is not empty then (3.8) gives the absolute estimate of the difference between optimal and quasioptimal solutions for all the vectors

$$\mathbf{g} = (g_1^{0\,1/\beta_1}, g_2^{0\,1/\beta_2}, \dots g_n^{0\,1/\beta_n}), \qquad \mathbf{g} \in G(\mathbf{k}).$$

Further, we may take the maximum of the estimate (3.8) for all the possible vectors $\mathbf{k}$, i.e. such vectors that the corresponding regions $G(\mathbf{k}) \neq \emptyset$, and hence achieve an absolute estimate of quasioptimal solution in the quadrant $g_i^{0\,1/\beta_i} > 0$, $i = 1, 2, \ldots n$.

Notice that the optimal solutions $x_i^0 = g_i^{0\,1/\beta_i} x_i^*$ are realized in the regions defined by the inequalities

$$g_j^{0\,1/\beta_j} / g_i^{0\,1/\beta_i} \geqslant g_j^{-1/\beta_j}(x_i^*); \qquad j = 1, 2, \ldots n,\ j \neq i. \tag{3.10}$$

In fact, by substitution it is easy to see that all the constraints of (3.1) are satisfied, the value $\gamma_i = 1$ and the quasioptimal solution in this region according to (3.8) coincides with the optimal $x_q = x_i^0$. This enables to exclude from the analysis all the vectors $\mathbf{k}$ (and corresponding regions $G(\mathbf{k})$) whose coordinate k_i equals i.

It should be also noted that for the analysis of the regions where optimal and quasioptimal solutions are realized it is convenient to introduce new variables, for example

$$t_i = g_i^{0\,1/\beta_i} / g_n^{0\,1/\beta_n}, \qquad i = 1, 2, \ldots n - 1,$$

and instead of the n-dimensional space of parameters

$$g_i^{0\,1/\beta_i}, \qquad i = 1, 2, \ldots n$$

pass to the $(n-1)$-dimensional space of t_i, $i = 1, 2, \ldots n - 1$.

4. EXAMPLE 2. BEAM OF MINIMUM WEIGHT WITH THREE DIFFERENT CONSTRAINTS

Statement. Consider the problem of determining the shape of an elastic beam of minimum weight under the action of three different constraints on the mechanical behaviour: first frequency of natural transverse vibrations, buckling load under action of axial force P and maximum deflection under action of bending load—transverse concentrated force Q applied at the centre of the beam span [8]. These three external influences are assumed to be applied to the structure separately and independently, i.e. their simultaneous action is not considered. Beam cross-section is assumed to be rectangular with constant breadth and variable thickness. Simply supported boundary conditions of the beam ends are considered.

The processes of a beam vibration, buckling and bending are governed by the equations written in dimensionless form

$$(h^3(x) w_1''(x))'' - \omega^2 h(x) w_1(x) = 0; \qquad w_1(0) = w_1(1) = h^3 w_1''|_{x=0} = h^3 w_1''|_{x=1} = 0 \tag{4.1}$$

$$h^3(x) w_2''(x) + p w_2(x) = 0; \qquad w_2(0) = w_2(1) = 0 \tag{4.2}$$

$$h^3(x) w_3''(x) - q m(x) = 0; \qquad w_3(0) = w_3(1) = 0 \tag{4.3}$$

$$m(x) = 3/4x, \quad 0 \leqslant x \leqslant 1/2; \qquad m(x) = 3/4(1-x), \quad 1/2 \leqslant x \leqslant 1;$$
$$\omega^2 = 3\Omega^2 \rho l^2 / 2E; \qquad p = 3p/2ECl; \qquad q = Q/ECl. \tag{4.4}$$

In these equations Ω is frequency of vibration, P is magnitude of buckling load, Q is magnitude of concentrated bending force; ω, p, q are corresponding non-dimensional quantities.

$$2h(x),\ w_1(x),\ w_2(x),\ w_3(x)$$

are non-dimensional quantities: the beam thickness and deflection functions at vibration, buckling and bending respectively, divided on the beam span l. E, ρ, C are: Young's modulus, mass density and beam breadth. Non-dimensional weight of a beam is described by the functional

$$v = \int_0^1 h(x)\,\mathrm{d}x. \tag{4.5}$$

For the sake of convenience we introduce $w = w_3(1/2)$.

Applying constraints on the first natural frequency, buckling load and deflection at the beam span centre when bending takes place for non-dimensional quantities we obtain

$$\omega^2 \geq \omega_0^2,\, p \geq p_0,\, w \leq w_0 \tag{4.6}$$

where ω_0, p_0, w_0 are non-dimensional given constants.

The optimization problem consists in determining the beam thickness $h^0(x) > 0$, $x \in (0, 1)$ so that the relations (4.1)–(4.3) and the restrictions (4.6) are satisfied.

Note that the problem of maximizing the first natural frequency of vibrations for beams with similar cross sections and fixed volume of material was solved numerically by Niordson[13], the analytical solutions of optimization problems of buckling columns were presented in the paper of Tadjbakhsh and Keller[14]. The solution of the optimization problem with only the third constraint can easily be obtained analytically.

Necessary optimality conditions. Because the differential operators considered are positive and self-adjoint to the eigenvalue problems (4.1) and (4.2) the Rayleigh principle may be applied

$$\omega^2(h) = \min_{u_1} \frac{\int_0^1 h^3 u_1''^2\, dx}{\int_0^1 h u_1^2\, dx} = \frac{\int_0^1 h^3 w_1''^2\, dx}{\int_0^1 h w_1^2\, dx} \tag{4.7}$$

$$p(h) = \min_{u_2} \frac{\int_0^1 u_2'^2\, dx}{\int_0^1 u_2^2 h^{-3}\, dx} = \frac{\int_0^1 w_2'^2\, dx}{\int_0^1 w_2^2 h^{-3}\, dx}. \tag{4.8}$$

Admissible functions u_1 and u_2 in these expressions must satisfy the kinematical boundary conditions

$$u_1(0) = u_1(1) = 0, \qquad u_2(0) = u_2(1) = 0$$

and must be differenciable[15].

Integrating eqn (4.3) with the use of boundary conditions we get

$$w(h) = q \int_0^1 f(s) h^{-3}(s)\, ds \tag{4.9}$$

$$f(s) = 3/8\, \bar{S}^2,\ 0 \leq \bar{S} \leq 1/2; \qquad f(\bar{S}) = 3/8(1 - \bar{S})^2,\ 1/2 \leq \bar{S} \leq 1.$$

Gradients of the functionals (4.7)–(4.9) are proportional to the following quantities[5, 8]

$$\psi_1 = 3h^2 w_1''^2 - \omega^2 w_1^2,\ \psi_2 = w_2^2 h^{-4},\ \psi_3 = -f h^{-4}.$$

Gradient of the functional (4.5) is equal to the unity. Using these relations we establish the necessary optimality conditions[16]

$$\begin{aligned} \mu_1\psi_1 + \mu_2\psi_2 + \mu_3\psi_3 &= 1, & x &\in (0, 1) \\ (\omega^2 - \omega_0^2)\mu_1 &= 0, & \mu_1 &\geq 0 \\ (p - p_0)\mu_2 &= 0, & \mu_2 &\geq 0 \\ (w - w_0)\mu_3 &= 0, & \mu_3 &\geq 0. \end{aligned} \tag{4.10}$$

When problem parameters ω_0, p_0, w_0, q are given the system of eqns (4.1)–(4.3), (4.10) enables to obtain the functions $h^0(x)$, $w_1^0(x)$, $w_2^0(x)$, $w_3^0(x)$ and the multipliers μ_1, μ_2, μ_3 which realize the optimal solution of the problem.

Depending on the relations of the problem parameters some of the constraints (4.6) may or may not be active, for example when $\omega = \omega_0$, $p > p_0$, $w < w_0$, then according to (4.10) we obtain $\mu_1 > 0$, $\mu_2 = \mu_3 = 0$. Therefore, for this case the problem considered reduces to the problem with the single first constraint, etc.

Optimal and quasioptimal solutions. Let us reformulate now the stated optimization problem in the terms of the previous Section 3.

$$\min v(h)$$

$$\omega^2(h) \geq \omega_0^2, \qquad p(h) \geq p_0, \qquad \frac{q}{w(h)} \geq \frac{q}{w_0} \tag{4.11}$$

$$h(x) > 0, \qquad x \in (0, 1).$$

From the expressions (4.5)–(4.9) it immediately follows that the functionals $v(h)$, $\omega^2(h)$, $p(h)$, $q/w(h)$ are positive when $h(x) > 0$ and homogeneous in h with the homogeneity degrees $\alpha = 1$, $\beta_1 = 2$, $\beta_2 = 3$, $\beta_3 = 3$, respectively.

The solutions of (4.11) with the single ith constraint may be taken in the form

$$h_1^0(x) = \omega_0 h_1^*(x), \qquad h_2^0(x) = p_0^{1/3} h_2^*(x), \qquad h_3^0(x) = (q/w_0)^{1/3} h_3^*(x). \tag{4.12}$$

The functions h_1^*, h_2^* were calculated by the gradient method in the space of control functions $h(x)$, and at every step of gradient procedure the eigenvalue problems (4.1), (4.2) were solved (more detailed about computations see [5]). The functions h_1^*, h_2^* are presented in the Fig. 1. In view of the symmetry of these functions only one half the span is shown.

The solution of (4.11) with the single third constraint can easily be obtained analytically

$$h_3^*(x) = \begin{cases} 2^{-5/6}\surd(x), & 0 \leq x \leq 1/2 \\ 2^{-5/6}\surd(1-x), & 1/2 \leq x \leq 1 \end{cases}.$$

This function has an angular point at $x = 1/2$. Having obtained the functions h_1^*, h_2^*, h_3^* we calculate the matrix, see (3.7)

$$\eta_{ij} = g_j^{-1/\beta_j}(x_i^*). \tag{4.13}$$

This matrix is presented in the Fig. 2. According to (3.10) the solutions $h_1^0(x)$, $h_2^0(x)$, $h_3^0(x)$ from (4.12) are realized in the regions of parameters I, II, III defined by the inequalities

$$g_i^{0\,1/\beta_i}/g_j^{0\,1/\beta_j} \geq g_j^{-1/\beta_j}(x_i^*), \qquad j = 1, 2, \ldots n;\ j \neq i.$$

From these we get

I
$$\frac{\omega_0}{p_0^{1/3}} \geq p^{-1/3}(h_1^*), \qquad \frac{\omega_0}{(q/w_0)^{1/3}} \geq \left[\frac{q}{w(h_1^*)}\right]^{-1/3}$$

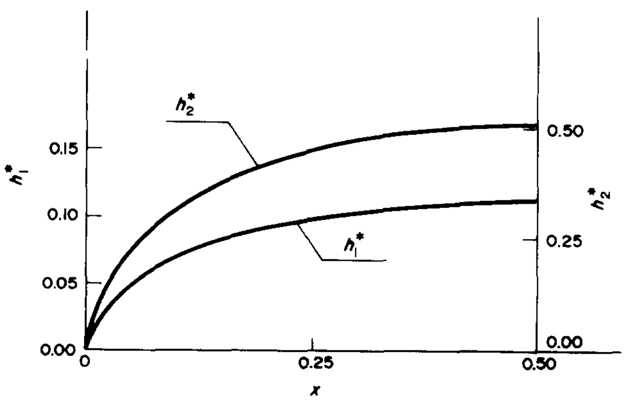

Fig. 1.

1.0	4.6043	2.9886
0.2174	1.0	0.6515
0.3513	1.6329	1.0

Fig. 2.

II $\qquad \dfrac{p_0^{1/3}}{\omega_0} \geq [\omega^2(h_2^*)]^{-1/2}, \quad \left(\dfrac{p_0 w_0}{q}\right)^{1/3} \geq \left[\dfrac{1}{w(h_2^*)}\right]^{-1/3}$

III $\qquad \dfrac{(q/w_0)^{1/3}}{\omega_0} \geq [\omega^2(h_3^*)]^{-1/2}, \quad \left(\dfrac{q}{w_0 p_0}\right)^{1/3} \geq [p(h_3^*)]^{-1/3}.$

For the sake of convience we introduce new parameters

$$t_1 = \left(\frac{q}{w_0}\right)^{1/3} \frac{1}{\omega_0},\ t_2 = \left(\frac{q}{w_0 p_0}\right)^{1/3},\ t_3 = \frac{\omega_0}{p_0^{1/3}} = \frac{t_2}{t_1}.$$

With the use of the new variables the inequalities defining the regions I, II, III take the form

$$\text{I } t_3 \geq a_1, t_1 \leq a_2 \qquad \text{II } t_3 \leq b_1, t_2 \leq b_2 \qquad \text{III } t_1 \geq c_1, t_2 \geq c_2$$

where

$$a_1 = 4.6043, a_2 = 0.3346, b_1 = 4.5993, b_2 = 1.5349, c_1 = 0.3513, c_2 = 1.6329.$$

The regions I, II, III are presented in the Fig. 3. So, if the problem parameters ω_0, p_0, w_0, q are such that the point $D(t_1, t_2)$ lies in the region I, II or III then the functions

$$h_1^0(x) = \omega_0 h_1^*(x), \quad h_2^0(x) = p_0^{1/3} h_2^*(x), \quad h_3^0(x) = (q/w_0)^{1/3} h_3^*(x)$$

will be the corresponding optimal solutions.

Consider now the case when the problem parameters are such that the corresponding point D does not belong to the indicated regions I, II, III. Then according to (3.2) one may calculate

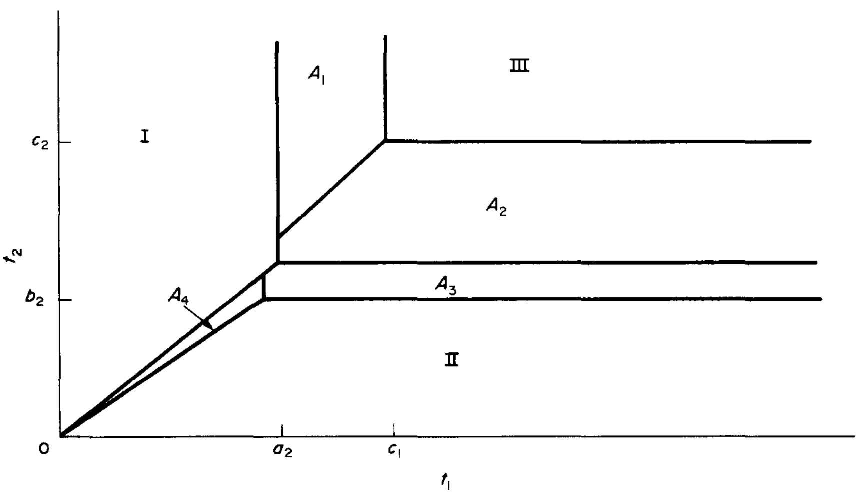

Fig. 3.

quantities γ_i, construct the quasioptimal solution (3.4) and from (3.5) obtain an estimate of closeness of this solution to the optimal.

To achieve absolute estimates (independent of the parameters ω_0, p_0, w_0, q) it is required to construct the regions $G(\mathbf{k})$ according to (3.9).

In this problem only the vectors $\mathbf{k}$ (3, 3, 1), (3, 3, 2), (2, 1, 2), (2, 3, 2) prove to be admissible, i.e. the corresponding regions $G(\mathbf{k})$ are non-empty. In the plane t_1, t_2 the inequalities (3.9) define the regions A_1, A_2, A_3, A_4, see Fig. 3.

$$A_1\colon a_2 \leqslant t_1 \leqslant c_1,\ t_2/t_1 \geqslant c_2/c_1$$

$$A_2\colon t_1 \geqslant a_2,\ t_2/t_1 \leqslant c_2/c_1,\ a_1a_2 \leqslant t_2 \leqslant c_2$$

$$A_3\colon t_1 \geqslant b_2/b_1,\ t_2/t_1 \leqslant a_1,\ b_2 \leqslant t_2 \leqslant a_1a_2$$

$$A_4\colon t_1 \leqslant b_2/b_1,\ b_1 \leqslant t_2/t_1 \leqslant a_1.$$

According to (3.8) for these regions we have the following absolute estimates

$$1 \leqslant \frac{v(h_q)}{v(h^0)} \leqslant \min\left[\left(\frac{q}{w(h_1^*)}\right)^{-1/3}\frac{v(h_1^*)}{v(h_3^*)}, \left(\frac{q}{w(h_2^*)}\right)^{-1/3}\frac{v(h_2^*)}{v(h_3^*)}, \frac{\omega^{-1}(h_3^*)v(h_3^*)}{v(h_1^*)}\right]$$

$$= \left(\frac{q}{w(h_1^*)}\right)^{-1/3}\frac{v(h_1^*)}{v(h_3^*)} = 1.021; \qquad D \in A_1$$

$$1 \leqslant \frac{v(h_q)}{v(h^0)} \leqslant \min\left[\left(\frac{q}{w(h_1^*)}\right)^{-1/3}\frac{v(h_1^*)}{v(h_3^*)}, \left(\frac{q}{w(h_2^*)}\right)^{-1/3}\frac{v(h_2^*)}{v(h_3^*)}, \frac{p^{-1/3}(h_3^*)v(h_3^*)}{v(h_2^*)}\right]$$

$$= \left(\frac{q}{w(h_1^*)}\right)^{-1/3}\frac{v(h_1^*)}{v(h_3^*)} = 1.021; \qquad D \in A_2$$

$$1 \leqslant \frac{v(h_q)}{v(h^0)} \leqslant \min\left[\frac{p^{-1/3}(h_1^*)v(h_1^*)}{v(h_2^*)}, \left(\frac{q}{w(h_3^*)}\right)^{-1/3}\frac{v(h_2^*)}{v(h_3^*)}, \frac{p^{-1/3}(h_3^*)v(h_3^*)}{v(h_2^*)}\right]$$

$$= \frac{p^{-1/3}(h_1^*)v(h_1^*)}{v(h_2^*)} = 1.00064; \qquad D \in A_3$$

$$1 \leqslant \frac{v(h_q)}{v(h^0)} \leqslant \min\left[\frac{p^{-1/3}(h_1^*)v(h_1^*)}{v(h_2^*)}, \frac{\omega^{-1}(h_2^*)v(h_2^*)}{v(h_1^*)}, \frac{p^{-1/3}(h_3^*)v(h_3^*)}{v(h_2^*)}\right]$$

$$= \frac{\omega^{-1}(h_2^*)v(h_2^*)}{v(h_1^*)} = 1.00046; \qquad D \in A_4.$$

Thus, for all non-negative values of parameters ω_0, p_0, w_0, q quasioptimal solutions in this problem exceed the optimal by the value of minimized functional v less than 2.1%.

Notice that for the structural optimization problems with multiple constraints the quasioptimal solutions may be constructed when the additional constraint on structural variable $h(x) \geqslant \epsilon_0 > 0$ is implied, but the procedure to obtain an absolute estimate would become more complicated.

CONCLUSIONS

On the basis of homogeneity properties of functionals, usually considered in structural optimization problems, it was possible to state the equivalence of the solutions of dual structural optimization problems and to construct the quasioptimal solutions with two-side estimations of minimized functional value for multi-purpose problems. The quasioptimization technique is simple and efficient because it enables us, with the use of optimal solutions with the single constraint, to approximate the optimal solution with complete set of constraints and to establish the error estimates independent of the problem parameters.

The method of quasioptimal solutions can be applied to multi-purpose structural optimization problems involving trusses, beams, arches and plates governed by linear differential equations with homogeneous boundary conditions. As the constraints various functionals may be considered-frequencies of natural vibrations, buckling loads, maximal values of stress, deflection or invariants of stress tensor under the action of static loads, etc. The quasioptimization method would be particularly useful for numerical solution of two-dimensional structural optimization problems with multiple constraints and parameters.

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